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The generalized geometry, equivariant $\bar{\partial}\partial$-lemma, and torus actions

Yi Lin*

Department of Mathematics, Bahen Center for Information, University of Toronto, Canada, M5S2E4

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Abstract

In this paper we first consider the Hamiltonian action of a compact connected Lie group on an H-twisted generalized complex manifold M. Given such an action, we define generalized equivariant cohomology and generalized equivariant Dolbeault cohomology. If the generalized complex manifold M satisfies the $\bar{\partial}\partial$-lemma, we prove that they are both canonically isomorphic to $(S\mathfrak{g}^*)^G \otimes H_H(M)$, where $(S\mathfrak{g}^*)^G$ is the space of invariant polynomials over the Lie algebra $\mathfrak{g}$ of G, and $H_H(M)$ is the H-twisted cohomology of M. Furthermore, we establish an equivariant version of the $\bar{\partial}\partial$-lemma, namely the $\bar{\partial}_G\partial$-lemma, which is a direct generalization of the $d_G\delta$-lemma [Y. Lin, R. Sjamaar, Equivariant symplectic Hodge theory and $d_G\delta$-lemma, J. Symplectic Geom. 2 (2) (2004) 267–278] for Hamiltonian symplectic manifolds with the Hard Lefschetz property.

Second we consider the torus action on a compact generalized Kähler manifold which preserves the generalized Kähler structure and which is equivariantly formal. We prove a generalization of a result of Carrell and Lieberman [J.B. Carrell, D.I. Lieberman, Holomorphic vector fields and compact Kähler manifolds, Invent. Math. 21 (1973) 303–309] in generalized Kähler geometry. We then use it to compute the generalized Hodge numbers for non-trivial examples of generalized Kähler structures on $\mathbb{CP}^n$ and $\mathbb{CP}^n$ blown up at a fixed point.

Keywords: Hamiltonian actions on generalized complex manifolds; Generalized Kähler manifolds; Generalized Hodge theory

1. Introduction

Generalized complex geometry, as introduced by Hitchin [17]and further developed by Gualtieri [10], provides a unifying framework for both symplectic and complex geometry. It is no surprise that many facts in complex geometry have their counterparts in generalized complex geometry. For instance, it is well known that a complex structure induces a (p, q)-decomposition for differential forms and a splitting $d = \partial + \bar{\partial}$. Analogously, Gualtieri [10] proved that the presence of an H-twisted generalized complex structure on a manifold determines an alternative grading of differential forms and a similar splitting $d_H = \partial + \bar{\partial}$, where H is a closed 3-form and $d_H = d - H\wedge$ is the twisted exterior derivative. (For the twisted case, see also the Appendix of [19].) Therefore it makes perfect sense to define the generalized Dolbeault cohomology and the $\bar{\partial}\partial$-lemma for H-twisted generalized complex manifolds. Namely, a twisted generalized complex manifold is said to satisfy the $\bar{\partial}\partial$-lemma if

$$\ker\partial \cap \operatorname{im}\bar{\partial} = \operatorname{im}\partial \cap \ker\bar{\partial} = \operatorname{im}\bar{\partial}\partial.$$

* Tel.: +1 416 978 5214; fax: +1 416 978 4107.
E-mail address: yilin@math.toronto.edu.

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Indeed, the $\bar{\partial}\partial$-lemma in generalized geometry has been studied extensively by Cavalcanti in his thesis [12]. It is interesting to study it for many reasons. When a generalized complex structure is induced by a symplectic structure, the $\bar{\partial}\partial$-lemma is equivalent to the Hard Lefschetz property, as established by Merkulov [27] and Guillemin [14]; whereas when a generalized complex structure is induced by a complex structure, the $\bar{\partial}\partial$-lemma coincides with the usual $\bar{\partial}\partial$-lemma in complex geometry, which is known to carry a lot of topological information. (See for instance [8].) Recently, Gualtieri [11] proved that a compact H-twisted generalized Kähler manifold satisfies the $\bar{\partial}\partial$-lemma with respect to both generalized complex structures involved. This result plays an important role in the remarkable works of [21,13] which assert that the moduli space of generalized complex structures on a compact H-twisted generalized Calabi–Yau manifold is unobstructed.

In this paper we consider the consequence of the $\bar{\partial}\partial$-lemma for group actions on generalized complex manifolds. Sources from both symplectic and complex geometry have served as motivations for this work.

In [22] Sjamaar and the author studied the Hamiltonian action of a compact connected Lie group on a symplectic manifold with the Hard Lefschetz property. The main results are an equivariant version of the symplectic $d\delta$-lemma, i.e., the $d_G\delta$-lemma, and a stronger version of Kirwan–Ginzburg formality theorem which says that each cohomology class has a *canonical* equivariant extension.

Motivated by [22], we consider the Hamiltonian action of a compact connected Lie group on an H-twisted generalized complex manifold $(M, \mathcal{J})$ as introduced in [25].[1] Given such an action, we introduce two extensions of the usual equivariant Cartan complex and define their cohomologies to be the generalized equivariant cohomology and the generalized equivariant Dolbeault cohomology respectively. In contrast with the usual equivariant Cartan complex, these two extensions both contain information from moment 1-forms which come up very naturally in the definition of generalized moment maps. Assume the manifold satisfies the $\bar{\partial}\partial$-lemma, we prove that the generalized equivariant cohomology and the generalized equivariant Dolbeault cohomology are both canonically isomorphic to $(S\mathfrak{g}^*)^G \otimes H_H(M)$, where $(S\mathfrak{g}^*)^G$ is the space of invariant polynomials over $\mathfrak{g}$ and $H_H(M)$ is the H-twisted cohomology of the manifold M. This gives an analogue of Kirwan–Ginzburg equivariant formality theorem in generalized geometry. In addition, we establish an equivariant version of the $\bar{\partial}\partial$-lemma, namely, the $\bar{\partial}_G\partial$-lemma, which is a direct generalization of the $d_G\delta$-lemma [22] in symplectic geometry.

We would like to mention that recently it has been found by Kapustin and Tomasiello [20] that the conditions used in [25] to define Hamiltonian actions and reductions in generalized Kähler geometry are exactly the same as the physics conditions for $(2, 2)$ gauged sigma model. This has thus provided us physics motivations to study the properties of the Hamiltonian generalized Kähler manifolds as defined in [25]. As a consequence of the results stated in the previous paragraph, one derives easily the first of such properties: compact Hamiltonian twisted generalized Kähler manifolds always satisfy the $\bar{\partial}_G\partial$-lemma. In view of this and the aforementioned result of A. Kapustin and A. Tomasiello, it will be interesting to construct more non-trivial examples of compact Hamiltonian generalized Kähler manifolds. This question has been addressed in an accompanying short note [24] which also provides us some interesting examples for which the $\bar{\partial}_G\partial$-lemma holds.

The second part of this paper is guided by results from Kähler geometry. Historically, holomorphic vector fields on Kähler manifolds have been studied by many mathematicians. Among many other things, a famous result of Carrell and Lieberman [6] asserts if on a compact Kähler manifold M there exists a holomorphic vector field which has only isolated zeros, then $H^{p,q}_{\bar{\partial}}(M) = 0$ unless $p = q$. Recently, assuming the holomorphic vector field is generated by a torus action, Carrell, Kaveh and Puppe [7] gave a new proof of this result. Their method is based on equivariant Dolbeault decomposition as recently treated by Teleman [29] and Lillywhite [26], as well as the *localization theorem* in equivariant cohomology theory.

We observe that the new treatment given in [7] could be adapted to the case of a torus action on a compact generalized Kähler manifold under certain conditions. Indeed, if we assume the action of the torus T is equivariantly formal, then a result of Allday [1] shows that the equivariant cohomology of the torus action is canonically isomorphic to $S \otimes H(M)$, where S is the space of polynomials over the Lie algebra of T. On the other hand, it is shown [12] that assuming the $\bar{\partial}\partial$-lemma $H(M)$ will split into the direct sum of generalized Dolbeault cohomology groups. Therefore we actually have a version of generalized equivariant Dolbeault decomposition for compact generalized

[1] We note that recently there have been considerable interests in extending quotient and reduction to the realm of generalized complex geometry [18,25,28,5,30].

Kähler manifolds. This result, together with the localization theorem in equivariant cohomology theory, enables us to get a generalization of the above mentioned result of Carrell and Lieberman in generalized Kähler geometry.

Actually, another motivation for this piece of work is to understand the generalized Hodge theory developed by Gualtieri [11] by some concrete examples. In [25] Tolman and the author developed a general method which allows one to produce non-trivial examples of generalized Kähler structures on many toric varieties. We note that the classical Hodge numbers have been long known for toric varieties. It is thus an interesting question if one can calculate the generalized Hodge numbers for the examples of generalized Kähler structures on toric varieties discovered in [25]. In this article, using the analogue of Carrell and Lieberman's result in generalized Kähler geometry we compute the generalized Hodge number for non-trivial examples of generalized Kähler structures on $\mathbb{CP}^n$ and $\mathbb{CP}^n$ blown up at a fixed point.

The plan of this paper is as follows.

Section 2 goes over some basic concepts in generalized geometry.

Section 3 presents a quick review of equivariant de Rham theory, including a recent result of Allday [1].

Section 4 defines the generalized equivariant cohomology and generalized equivariant Dolbeault cohomology for Hamiltonian actions on twisted generalized complex manifolds. Assume that the manifold M has the $\bar\partial\partial$-lemma, Section 4 proves that the two cohomologies are canonically isomorphic to $(S\mathfrak{g}^*)^G \otimes H_H(M)$; moreover, it establishes an equivariant version of the $\bar\partial\partial$-lemma.

Section 5 presents a generalization of Carrell and Lieberman's result in generalized Kähler geometry.

Section 6 computes the generalized Hodge number for non-trivial examples of generalized Kähler structures on $\mathbb{CP}^n$ and $\mathbb{CP}^n$ blown up at a point.

2. Generalized complex geometry

Let V be an n-dimensional vector space. There is a natural bi-linear pairing of type (n, n) on $V \oplus V^*$ which is defined by

$$\langle X + \alpha, Y + \beta\rangle = \frac{1}{2}(\beta(X) + \alpha(Y)).$$

A *generalized complex structure* on a vector space V is an orthogonal linear map $\mathcal{J} : V \oplus V^* \to V \oplus V^*$ such that $\mathcal{J}^2 = -1$. Let $V \subset V_{\mathbb{C}} \oplus V^*_{\mathbb{C}}$ be the $\sqrt{-1}$ eigenspace of the generalized complex structure $\mathcal{J}$. Then L is maximal isotropic and $L \cap \overline{L} = \{0\}$. Conversely, given a maximal isotropic subspace $L \subset V_{\mathbb{C}} \oplus V^*_{\mathbb{C}}$ so that $L \cap \overline{L} = \{0\}$, there exists an unique generalized complex structure $\mathcal{J}$ whose $\sqrt{-1}$ eigenspace is exactly L.

Let $\pi : V_{\mathbb{C}} \oplus V^*_{\mathbb{C}} \to V_{\mathbb{C}}$ be the natural projection. The *type* of $\mathcal{J}$ is the codimension of $\pi(L)$ in $V_{\mathbb{C}}$, where L is the $\sqrt{-1}$ eigenspace of $\mathcal{J}$.

The Clifford algebra of $V_{\mathbb{C}} \oplus V^*_{\mathbb{C}}$ acts on the space of forms $\wedge V^*$ via

$$(X + \xi) \cdot \alpha = \iota_X\alpha + \xi \wedge \alpha.$$

Since $\mathcal{J}$ is skew adjoint with respect to the natural pairing on $V \oplus V^*$, $\mathcal{J} \in so(V \oplus V^*) \cong \wedge^2(V \oplus V^*) \subset CL(V \oplus V^*)$. Therefore there is a Clifford action of $\mathcal{J}$ on the space of forms $\wedge V^*$ which determines an alternative grading: $\wedge V^*_{\mathbb{C}} = \bigoplus U^k$, where U^k is the $-k\sqrt{-1}$ eigenspace of the Clifford action of $\mathcal{J}$.

Let M be a manifold of dimension n. There is a natural pairing of type (n, n) which is defined on $TM \oplus T^*M$ by

$$\langle X + \alpha, Y + \beta\rangle = \frac{1}{2}(\beta(Y) + \alpha(X))$$

and which extends naturally to $T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$.

For a closed 3-form H, the *H-twisted Courant bracket* of $T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ is defined by the identity

$$[X + \xi, Y + \eta] = [X, Y] + L_X\eta - L_Y\xi - \frac{1}{2}d(\eta(X) - \xi(Y)) + \iota_Y\iota_X H.$$

The Clifford algebra of $C^\infty(TM \oplus T^*M)$ with the natural pairing acts on differential forms by

$$(X + \xi) \cdot \alpha = \iota_X\alpha + \xi \wedge \alpha.$$

A *generalized almost complex structure* on a manifold M is an orthogonal bundle map $\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M$ such that $\mathcal{J}^2 = -1$. Moreover, $\mathcal{J}$ is an *H-twisted generalized complex structure* if the sections of the $\sqrt{-1}$ eigenbundle of $\mathcal{J}$ is closed under the H-twisted Courant bracket. The type of $\mathcal{J}$ at $m \in M$ is the *type* of the restricted generalized complex structure on $T_m M$.

Let B be a closed 2-form on a manifold M, and consider the orthogonal bundle map $TM \oplus T^*M \to TM \oplus T^*M$ defined by

$$e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix},$$

where B is regarded as a skew-symmetric map from TM to T^*M. This map preserves the H-twisted Courant bracket. As a simple consequence, if $\mathcal{J}$ is an H-twisted generalized complex structure on M, then $\mathcal{J}' = e^B \mathcal{J} e^{-B}$ is another H-twisted generalized complex structure on M, called the B-transform of $\mathcal{J}$. Moreover, the $\sqrt{-1}$ eigenbundle of $\mathcal{J}'$ is $e^B(L)$, so $\mathcal{J}$ and $\mathcal{J}'$ have the same type.

Let $(M, \mathcal{J})$ be an H-twisted generalized complex manifold of dimension $2n$, and let L be the $\sqrt{-1}$ eigenbundle of $\mathcal{J}$. Since $\mathcal{J}$ can be identified with a smooth section of the Clifford bundle $\mathrm{CL}(TM \oplus T^*M)$, there is a Clifford action of $\mathcal{J}$ on the space of differential forms. Let U^k be the $-k\sqrt{-1}$ eigenbundle of $\mathcal{J}$. [10] shows that there is a grading of the differential forms:

$$\Omega^*(M) = \Gamma(U^{-n}) \oplus \cdots \oplus \Gamma(U^0) \oplus \cdots \oplus \Gamma(U^n);$$

moreover, Clifford multiplication by sections of L and $\overline{L}$ is of degree -1 and 1 respectively with respect to this grading. This elementary fact plays an important role in Section 4.

It has been shown (See e.g. [10,19]) that the integrability of an H-twisted generalized complex structure $\mathcal{J}$ implies that

$$d_H = d - H\wedge : \Gamma(U^k) \to \Gamma(U^{k-1}) \oplus \Gamma(U^{k+1}),$$

which gives rise to operators ∂ and $\bar{\partial}$ via the projections

$$\partial : \Gamma(U^k) \to \Gamma(U^{k-1}), \qquad \bar{\partial} : \Gamma(U^k) \to \Gamma(U^{k+1}).$$

It follows that

$$\bar{\partial}^2 = \partial^2 = \bar{\partial}\partial + \partial\bar{\partial} = 0.$$

This leads to the following definition.

Definition 2.1 (*Cf. [12]*). The k-th *generalized Dolbeault cohomology* of M is defined to be

$$H^k_{\bar{\partial}}(M) = \ker\,(\Gamma(U^k) \xrightarrow{\bar{\partial}} \Gamma(U^{k+1}))/\mathrm{im}\,(\Gamma(U^{k-1}) \xrightarrow{\bar{\partial}} \Gamma(U^k)).$$

The effect of B-transforms on the above grading of differential forms and on $\bar{\partial}$, ∂ operators has been studied in [12]. Though [12] considers only the untwisted generalized complex structures, it is easily seen that the same proof extends to the twisted case as well. However, note that our sign convention for B-transforms differs from that of [12].

Lemma 2.2 (*[12]*). *Let B be a closed 2-form and let $\mathcal{J}_B$ be the B-transform of the generalized complex structure $\mathcal{J}$. Then we have*
(a) $U^k_B = e^{-B} U^k$, *where U^k_B denotes the $-k\sqrt{-1}$ eigenbundle of $\mathcal{J}_B$;*
(b) $\bar{\partial}_B = e^{-B}\bar{\partial}e^B$, $\partial_B = e^{-B}\partial e^B$.

Proposition 2.3 (*[12]*). *If the generalized complex manifold $(M, \mathcal{J})$ satisfies the $\bar{\partial}\partial$-lemma, then*

$$H_H(M) = \bigoplus_k H^k_{\bar{\partial}}(M).$$

Example 2.4 (*[10]*).

- Let V be a real vector space with a complex structure I. Then the map $\mathcal{J} : V \oplus V^* \to V \oplus V^*$ defined by

$$\mathcal{J}=\begin{pmatrix}-I^* & 0\\ 0 & I\end{pmatrix} \tag{2.1}$$

is a generalized complex structure on V with the $\sqrt{-1}$ eigenspace $L=\wedge^{0,1}V_{\mathbb{C}}\oplus\wedge^{1,0}V^*_{\mathbb{C}}$. And one easily checks that $U^k=\bigoplus_{q-p=k}(\wedge^{p,q}V^*_{\mathbb{C}})$.

- Now let (M,I) be a complex manifold. Then (2.1) defines a generalized complex structure with the $\sqrt{-1}$ eigenbundle

$$L=\wedge^{0,1}T_{\mathbb{C}}M\oplus\wedge^{1,0}T^*_{\mathbb{C}}M.$$

And the decomposition $d:\Gamma(U^k)\to\Gamma(U^{k+1})\oplus\Gamma(U^{k-1})$ coincides with the usual decomposition $d=\bar\partial+\partial$ on the complex manifold (M,I).

Example 2.5 (*[12]*).

- Let (V,ω) be a $2n$-dimensional symplectic vector space. Then the map $\mathcal{J}:V\oplus V^*\to V\oplus V^*$ defined by

$$\mathcal{J}=\begin{pmatrix}0 & -\omega^{-1}\\ \omega & 0\end{pmatrix} \tag{2.2}$$

is a generalized complex structure on V. It was proved in [12] that

$$U^k=\{e^{i\omega}e^{\frac{\wedge}{2i}}\alpha:\alpha\in\wedge^{n+k}V^*\}. \tag{2.3}$$

- Now let (M,ω) be a symplectic manifold. Then (2.2) defines a generalized complex structure $\mathcal{J}_\omega$ with the $\sqrt{-1}$ eigenbundle $L=\{X-\sqrt{-1}\iota_X\omega : X\in T_{\mathbb{C}}M\}$. It is easy to see that (2.3) provides a concrete description of the alternative grading of differential forms induced by $\mathcal{J}_\omega$. Furthermore, we have

$$-2i\partial(e^{i\omega}e^{\frac{\wedge}{2i}}\alpha)=e^{i\omega}e^{\frac{\wedge}{2i}}(\delta\alpha),\qquad \bar\partial(e^{i\omega}e^{\frac{\wedge}{2i}}\alpha)=e^{i\omega}e^{\frac{\wedge}{2i}}(d\alpha), \tag{2.4}$$

where δ is the Koszul's boundary operator introduced by Koszul [16] and studied by Brylinski [4]. As a consequence, the k-th generalized Dolbeault cohomology $H^k_{\bar\partial}(M)=H^{n-k}(M)$ as graded vector spaces.

Let $\Omega_\partial(M)=\Omega(M)\cap\ker\partial$. Since $\bar\partial$ anti-commutes with ∂, $(\Omega_\partial,\bar\partial)$ is a differential complex with the differential $\bar\partial$. Similarly, let $H(\Omega(M),\partial)$ be the homology of $\Omega(M)$ with respect to ∂. Then $\bar\partial$ induces a differential on $H(\Omega(M),\partial)$.

The following result is a simple consequence of the $\bar\partial\partial$-lemma. The proof is left as an exercise. (cf. [12].)

Proposition 2.6. *Assume that the $\bar\partial\partial$-lemma holds. Then the $\bar\partial$-chain maps in the diagram*

$$(\Omega(M),\bar\partial)\leftarrow(\Omega_\partial(M),\bar\partial)\to H(\Omega(M),\partial)$$

are quasi-isomorphisms, i.e., they induce isomorphisms in cohomology.

A manifold M is said to be an H-twisted generalized Kähler manifold if it has two commuting H-twisted generalized complex structures $\mathcal{J}_1$, $\mathcal{J}_2$ such that $\langle-\mathcal{J}_1\mathcal{J}_2\xi,\xi\rangle>0$ for any $\xi\neq 0\in C^\infty(T_{\mathbb{C}}M\oplus T^*_{\mathbb{C}}M)$, where $\langle\cdot,\cdot\rangle$ is the canonical pairing on $T_{\mathbb{C}}M\oplus T^*_{\mathbb{C}}M$. The following remarkable result is due to Gualtieri.

Theorem 2.7 (*[11]*). *Assume that $(M,\mathcal{J}_1,\mathcal{J}_2)$ is a compact H-twisted generalized Kähler manifold. Then it satisfies the $\bar\partial\partial$-lemma with respect to both $\mathcal{J}_1$ and $\mathcal{J}_2$.*

Example 2.8 (*[10]*).

- Let (ω,I) be a *genuine Kähler structure* on a manifold M, that is, a symplectic structure ω and a complex structure J which are *compatible*, which means that $g=-\omega J$ is a Riemannian metric. By Examples 2.4 and 2.5 ω and I induce generalized complex structures $\mathcal{J}_\omega$ and $\mathcal{J}_I$, respectively. Moreover, it is easy to see that $\mathcal{J}_\omega$ and $\mathcal{J}_I$ commute, and that

$$-\mathcal{J}_\omega\mathcal{J}_I=\begin{pmatrix}0 & g^{-1}\\ g & 0\end{pmatrix} \tag{2.5}$$

is a positive definite metric on $TM\oplus T^*M$. Hence $(\mathcal{J}_\omega,\mathcal{J}_I)$ is a generalized Kähler structure on M. Since $(\mathcal{J}_\omega,\mathcal{J}_I)$ is induced by a genuine Kähler structure, sometimes we will also call $(\mathcal{J}_\omega,\mathcal{J}_I)$ a Kähler structure.

- Let $(\mathcal{J}_\omega, \mathcal{J}_I)$ be a generalized Kähler structure induced by a genuine Kähler structure (ω, I), and let B be a closed 2-form. Then $(e^B(\mathcal{J}_\omega), e^B(\mathcal{J}_I))$ is also a generalized Kähler structure which is said to the B-transform of the Kähler structure (ω, I).

3. Equivariant de Rham theory and canonical equivariant extensions via Hodge theory

We begin with a rapid review of equivariant de Rham theory and refer to [15] for a detailed account. Let G be a compact connected Lie group and let $\Omega_G(M) = (S\mathfrak{g}^* \otimes \Omega(M))^G$ be the Cartan complex of the G-manifold M. For brevity we will write $\Omega = \Omega(M)$ and $\Omega_G = \Omega_G(M)$. By definition an element of Ω_G is an equivariant polynomial from $\mathfrak{g}$ to Ω and is called an equivariant differential form on M. The bi-grading of the Cartan complex is defined by $\Omega_G^{ij} = (S^i\mathfrak{g}^* \otimes \Omega^j)^G$. It is equipped with a vertical differential $1 \otimes d$, which is usually abbreviated to d, and the horizontal differential d', which is defined by $d'\alpha(\xi) = -\iota_\xi\alpha(\xi)$. Here ι_ξ denotes inner product with the vector field on M induced by $\xi \in \mathfrak{g}$. As a total complex, Ω_G has the grading $\Omega_G^k = \bigoplus_{2i+j=k} \Omega^{ij}$ and the total differential $d_G = d + d'$. The total cohomology $\ker d_G/\mathrm{im}\, d_G$ is the de Rham equivariant cohomology $H_G(M)$. A fundamental fact for equivariant cohomology is the following localization theorem.

Theorem 3.1 (*Localization Theorem*). *Suppose a compact connected torus T acts on a compact manifold M. Then the kernel of the canonical map*

$$i^* : H_T^*(M) \to H_T^*(M^T),$$

induced by the inclusion $i : M^T \to M$ is the module of torsion elements in $H_T(M)$, where M^T is the fixed point set of the torus T action. In particular, if $H_T(M)$ is a free module over S, the polynomial ring over the Lie algebra $\mathfrak{t}$ of T, then i^* is an injective map.

Since $\Omega_G^{0j} = (\Omega^G)^j$, the space of invariant j-forms on M, the zeroth column of the Cartan complex Ω_G is the invariant de Rham complex Ω^G. Because G is connected, Ω^G is a deformation retract of the ordinary de Rham complex Ω. The projection $\overline{p} : \Omega_G \to \Omega^G$, defined by $\overline{p}(\alpha) = \alpha(0)$, is a morphism between the Cartan complex (Ω_G, d_G) and the ordinary de Rham complex (Ω, d). The action of G is called *equivariantly formal* if $\overline{p}$ induces a surjective map $H_G(M) \to H(M)$. [1] explained in details that this definition of equivariant formality is equivalent to the one that the spectral sequence of the Cartan double complex relative to the row filtration degenerates at E_1 stage.

Assume the G action on M is equivariantly formal. Allday [1] showed how to construct canonical equivariant extensions using Hodge theory. Let us briefly recall his construction here. Using a G-invariant Riemannian metric on M one defines Hodge star operator $*$, adjoint operator d^*, the Laplacian $\triangle$, and Green's operator G. (It should be clear from the context where G is the Lie group and where G is Green's operator.) Since the metric is G-invariant, $*$, d^*, $\triangle$, and G are all G-equivariant operators. Therefore $P = (1 \otimes d^*G)d'$ is a well-defined operator on the equivariant Cartan complex. The following result is due to Allday.

Theorem 3.2 (*[1]*). *Assume the G-action on M is equivariantly formal. Let $\alpha \in \Omega^G$ be a closed form (i.e., $d\alpha = 0$). Let*

$$\widehat{\alpha} = (1 - P)^{-1}\alpha = \alpha + P(\alpha) + P^2(\alpha) + \cdots + P^n(\alpha) + \cdots.$$

Then $d_G\alpha = 0$. Hence the map $\alpha \to [\widehat{\alpha}]_G$, restricted to harmonic forms, is a canonical section of the projection $H_G(M) \to H(M)$.

As a direct consequence we have

Proposition 3.3. *Assume the G-action on M is equivariantly formal. Then there is a canonical isomorphism*

$$H_G(M) \cong (S\mathfrak{g}^*)^G \otimes H(M).$$

4. Generalized equivariant cohomology and the $\bar{\partial}_G \partial$-lemma

First we recall the definition of Hamiltonian actions on H-twisted generalized complex manifolds.

Definition 4.1 ([25]). Let a compact Lie group G with Lie algebra $\mathfrak{g}$ act on a manifold M, preserving an H-twisted generalized complex structure $\mathcal{J}$, where $H \in \Omega^3(M)^G$ is closed. Let $L \subset T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ denote the $\sqrt{-1}$ eigenbundle of $\mathcal{J}$. A *generalized moment map* is a smooth function $f: M \to \mathfrak{g}^*$ so that

- There exists a 1-form $\eta \in \Omega^1(M, \mathfrak{g}^*)$, called the *moment 1-form*, so that $\xi_M - \sqrt{-1}\,(\mathrm{d}f^\xi + \sqrt{-1}\,\eta^\xi)$ lies in $C^\infty(L)$ for all $\xi \in \mathfrak{g}$, where ξ_M denote the induced vector field.
- f is equivariant.
- $\iota_{\xi_M} H = \mathrm{d}\eta^\xi$ for any $\xi \in \mathfrak{g}$.

And the action of G is said to be Hamiltonian if such a generalized moment map exists.

Let Ω_G be the Cartan double complex of the G-manifold M. Then the horizontal differential of the Cartan complex is defined by $d'\alpha = -\iota_{\xi_M}\alpha(\xi)$, and the vertical differential is d. By definition, the Cartan double complex does not encode any information from the moment 1-form η which comes up very naturally in the definitions of generalized moment maps. Observe that $\iota_{\xi_M}\alpha = \xi_M \cdot \alpha$, where $\cdot$ denotes the spin action. To extract the full information from Definition 4.1, it is thus reasonable to extend d' to a new operator $\mathcal{A}$ by

$$(\mathcal{A}\alpha)(\xi) = \mathcal{A}(\xi)\cdot\alpha(\xi) = -\iota_{\xi_M}\alpha + \sqrt{-1}(\mathrm{d}f^\xi + \sqrt{-1}\eta^\xi)\wedge\alpha,$$

where $\mathcal{A}(\xi) = -\xi_M + \sqrt{-1}(\mathrm{d}f^\xi + \sqrt{-1}\eta^\xi)$. And now that the generalized complex manifold is H-twisted, it is natural to replace the usual derivative d by the twisted one $d_H = d - H\wedge$. Since H is an invariant form, the twisted exterior derivative d_H is G-equivariant and induces a well-defined operator $1 \otimes d_H$ on $(S\mathfrak{g}^* \otimes \Omega(M))^G$. For brevity let us also denote this by d_H. Then we have

$$\begin{aligned}
(d_H\mathcal{A}\alpha)(\xi) &= d_H\left(-\iota_{\xi_M}\alpha(\xi) + \sqrt{-1}\mathrm{d}f^\xi\wedge\alpha(\xi) - \eta^\xi\wedge\alpha(\xi)\right)\\
&= \left(-\mathrm{d}\iota_{\xi_M}\alpha(\xi) + H\wedge\iota_{\xi_M}\alpha(\xi)\right) - \sqrt{-1}\mathrm{d}f^\xi\wedge(d\alpha(\xi) - H\wedge\alpha(\xi))\\
&\quad + \eta^\xi\wedge(\mathrm{d}\alpha(\xi) - H\wedge\alpha(\xi)) - \mathrm{d}\eta^\xi\wedge\alpha(\xi)\\
&= \iota_{\xi_M}\mathrm{d}\alpha(\xi) - L_{\xi_M}\alpha(\xi) + H\wedge\iota_{\xi_M}\alpha(\xi) - \sqrt{-1}\mathrm{d}f^\xi\wedge d_H\alpha(\xi) + \eta^\xi\wedge d_H\alpha(\xi)\\
&\quad - \iota_{\xi_M}H\wedge\alpha(\xi) \quad (\text{Because } \mathrm{d}\eta^\xi = \iota_{\xi_M}H.)\\
&= (\xi_M - \sqrt{-1}\mathrm{d}f^\xi + \eta^\xi)\cdot d_H\alpha(\xi) \quad (\text{Because } L_{\xi_M}\alpha(\xi) = 0.)\\
&= (-\mathcal{A}d_H\alpha)(\xi).
\end{aligned}$$

This shows clearly that $d_H\mathcal{A} = -\mathcal{A}d_H$. We propose the following definition.

Definition 4.2 (*Generalized Equivariant Cohomology*). Let $\Omega_G = (S\mathfrak{g}^*\otimes\Omega(M))^G$ be Z_2 graded. Then $D_G = d_H + \mathcal{A}$ is a differential of degree 1. And the Z_2 graded generalized equivariant cohomology is defined to be

$$H^{\text{even/odd}}(\Omega_G, D_G) = \frac{\ker\left(\Omega_G^{\text{even/odd}} \overset{D_G}{\to} \Omega_G^{\text{odd/even}}\right)}{\operatorname{im}\left(\Omega_G^{\text{odd/even}} \overset{D_G}{\to} \Omega_G^{\text{even/odd}}\right)}.$$

As we are going to show in Example 4.5, the generalized equivariant cohomology is invariant under G-invariant B-transforms. This suggests that the generalized equivariant cohomology is something natural to work with in the category of generalized geometry. It will be interesting to define it for more general actions and study its property in some depth. We will leave it for future work.

The presence of the H-twisted generalized complex structure determines a splitting $d_H = \bar{\partial} + \partial$. And since $\mathcal{J}$ is G invariant, the operators $\bar{\partial}$ and ∂ are G equivariant. So on $(S\mathfrak{g}^*\otimes\Omega(M))^G$ there are well-defined operators $1\otimes\bar{\partial}$ and $1\otimes\partial$ which we will abbreviate to $\bar{\partial}$ and ∂. The following lemma says that $\bar{\partial}$ and ∂ also anti-commute with the operator $\mathcal{A}$ we introduced above.

Lemma 4.3. *For any* $\alpha \in (S\mathfrak{g}^* \otimes \Omega)^G$, *we have*

$$\bar{\partial}\mathcal{A}\alpha = -\mathcal{A}\bar{\partial}\alpha, \qquad \partial\mathcal{A}\alpha = -\mathcal{A}\partial\alpha.$$

Proof. Without the loss of generality we may assume that for any $\xi \in \mathfrak{g}$, $\alpha(\xi) \in U^k$, the $-k\sqrt{-1}$ eigenspace of the Clifford action of $\mathcal{J}$ on the space of differential forms. Observe that $(d_H\mathcal{A})\alpha(\xi) = -(\mathcal{A}d_H)\alpha(\xi)$, i.e., $d_H\mathcal{A}(\xi)\alpha(\xi) = -\mathcal{A}(\xi)d_H\alpha(\xi)$ for any $\xi \in \mathfrak{g}$. Compare the U^{k+1} and U^{k-1} components of $d_H\mathcal{A}(\xi)\alpha(\xi)$ and $-\mathcal{A}(\xi)d_H\alpha(\xi)$ respectively, we conclude that for any $\xi \in \mathfrak{g}$, $\mathcal{A}(\xi)\bar{\partial}\alpha(\xi) = -\bar{\partial}\alpha(\xi)$, $\mathcal{A}(\xi)\partial\alpha(\xi) = -\partial\alpha(\xi)$. □

Definition 4.4 (*Generalized Equivariant Dolbeault Cohomology*). Define $U_G = (S\mathfrak{g}^* \otimes \Omega(M))^G$ to be the double complex with the bi-grading

$$U_G^{i,j} = (S^i\mathfrak{g}^* \otimes \Gamma(U^{j-i}))^G,$$

where $\Gamma(U^{j-i})$ is the $(i-j)\sqrt{-1}$ eigenspace of the Clifford action of $\mathcal{J}$ on the space of differential forms. It is equipped with the vertical differential $\bar{\partial}$ and the horizontal differential $\mathcal{A}$. As a total complex, U_G has the grading $U_G^k = \bigoplus_{i+j=k} U_G^{i,j}$ and the total differential $\bar{\partial}_G = \bar{\partial} + \mathcal{A}$.The cohomology of the total complex $(U_G, \bar{\partial}_G)$ is defined to be the generalized equivariant Dolbeault cohomology of the Hamiltonian action.

Example 4.5. Let G act on an H-twisted generalized complex manifold $(M, \mathcal{J})$ with generalized moment map f and moment 1-form α. If $B \in \Omega^2(M)^G$, then G acts on the B-transform of $\mathcal{J}$ with generalized moment map f and moment 1-form α', where $(\alpha')^\xi = \alpha^\xi + \iota_{\xi_M}B$ for all $\xi \in \mathfrak{g}$.

- Let D_G^B be the total differential of the generalized equivariant double complex with respect to the Hamiltonian action on $(M, \mathrm{e}^B\mathcal{J}\mathrm{e}^{-B})$. Then a direct calculation shows that $D_G^B\alpha(\xi) = D_G\alpha(\xi) + (\iota_{\xi_M}B) \wedge \alpha(\xi)$. Since B is G-invariant, it induces an isomorphism

  $$e^B : \Omega_G \to \Omega_G, \qquad \alpha \mapsto e^B \wedge \alpha;$$

  furthermore, we have $D_G e^B = e^B D_G^B$. This shows immediately that

  $$H(\Omega_G, D_G) \cong H(\Omega_G, D_G^B).$$

- Let $\bar{\partial}_G^B$ be the total differential of the generalized equivariant Dolbeault complex with respect to the Hamiltonian G-action on $(M, e^B\mathcal{J}\mathrm{e}^{-B})$. It follows easily from Lemma 2.2 that $\bar{\partial}_G e^B = e^B\bar{\partial}_G^B$. As a result,

  $$H(U_G, \bar{\partial}_G^B) \cong H(U_G, \bar{\partial}_G).$$

Since by Lemma 4.3 the operator $\mathcal{A}$ anti-commute with $\partial : U_G^{i,j} \to U_G^{i,j-1}$, $\bar{\partial}_G = \bar{\partial} + \mathcal{A}$ anti-commutes with ∂. So it makes sense to define the $\bar{\partial}_G\partial$-lemma. Namely, the Hamiltonian generalized complex manifold M is said to satisfy the $\bar{\partial}_G\partial$-lemma if and only if

$$\ker\partial \cap \operatorname{im}\bar{\partial}_G = \ker\bar{\partial}_G \cap \operatorname{im}\partial = \operatorname{im}\bar{\partial}_G\partial.$$

First let us pause for a moment to point out that when the generalized complex structure is induced by a symplectic structure, the $\bar{\partial}_G\partial$-lemma is equivalent to the $d_G\delta$-lemma [22].

Example 4.6. Let G act on a symplectic manifold (M, ω) with moment map $\Phi : M \to \mathfrak{g}^*$, that is, Φ is equivariant and $\iota_{\xi_M}\omega = \mathrm{d}\Phi^\xi$ for all $\xi \in \mathfrak{g}$. Then G also preserves the generalized complex structure $\mathcal{J}_\omega$, i.e., the generalized complex structure induced by the symplectic structure ω, and Φ is a generalized moment map for this action with zero moment 1-form.

Since the symplectic structure ω and the associated Poisson bi-vector $\wedge$ are G-invariant, the operator $\mathrm{e}^{\mathrm{i}\omega}\mathrm{e}^{\frac{\wedge}{2\mathrm{i}}} : \Omega(M) \to \Omega(M)$ extends to an operator $\mathrm{e}^{\mathrm{i}\omega}\mathrm{e}^{\frac{\wedge}{2\mathrm{i}}} : (S\mathfrak{g}^* \otimes \Omega(M))^G \to U_G$. Moreover, for any equivariant differential form α and any $\xi \in \mathfrak{g}$, we have:

$$\begin{aligned}(\mathcal{A}\mathrm{e}^{\mathrm{i}\omega}\mathrm{e}^{\frac{\wedge}{2\mathrm{i}}}\alpha)(\xi) &= \mathcal{A}(\xi)\mathrm{e}^{\mathrm{i}\omega}\mathrm{e}^{\frac{\wedge}{2\mathrm{i}}}\alpha(\xi)\\ &= \mathrm{e}^{\mathrm{i}\omega}\iota_{\xi_M}\mathrm{e}^{\frac{\wedge}{2\mathrm{i}}}\alpha(\xi)\\ &= \mathrm{e}^{\mathrm{i}\omega}\mathrm{e}^{\frac{\wedge}{2\mathrm{i}}}\iota_{\xi_M}\alpha(\xi)\\ &= (\mathrm{e}^{\mathrm{i}\omega}\mathrm{e}^{\frac{\wedge}{2\mathrm{i}}}d'\alpha)(\xi).\end{aligned}$$

This proves that for any equivariant differential form α,

$$\mathcal{A}e^{i\omega}e^{\frac{\wedge}{2i}}\alpha = e^{i\omega}e^{\frac{\wedge}{2i}}d'\alpha.$$

This observation, together with (2.4), shows that

$$\bar{\partial}_G e^{i\omega}e^{\frac{\wedge}{2i}}\alpha = e^{i\omega}e^{\frac{\wedge}{2i}}d_G\alpha, \qquad -2i\partial e^{i\omega}e^{\frac{\wedge}{2i}}\alpha = e^{i\omega}e^{\frac{\wedge}{2i}}\delta\alpha,$$

where δ denotes the natural extension of the Koszul's boundary operator to equivariant differential forms.

Therefore the generalized equivariant cohomology group is canonically isomorphic to the equivariant cohomology group as $(S\mathfrak{g}^*)^G$-modules. Furthermore, it is easy to see that the $\bar{\partial}_G\partial$-lemma is equivalent to the $d_G\delta$-lemma [22] which asserts that

$$\ker d_G \cap \operatorname{im} \delta = \ker \delta \cap \operatorname{im} d_G = \operatorname{im} d_G\delta.$$

Now that $\bar{\partial}$ anti-commutes with ∂, it is straightforward to check that $U_{G,\partial} = U_G \cap \ker(\partial)$ is a sub-double complex of U_G and that the homology complex $H(U_G, \partial)$ of U_G with respect to ∂ is a double complex with differentials induced by $\bar{\partial}$ and $\mathcal{A}$. Thus we have the following diagram of morphisms of double complexes

$$(U_G, \bar{\partial}_G) \leftarrow (U_G \cap \ker(\partial), \bar{\partial}_G) \rightarrow H(U_G, \partial). \tag{4.1}$$

Since ∂ does not act on the polynomial part, these morphisms are linear over the invariant polynomials $(S\mathfrak{g}^*)^G$. Let us first examine the homology complex $H(U_G, \partial)$. We will need two preliminary results.

Lemma 4.7. *Suppose the action of the connected compact Lie group G on M preserves the H-twisted generalized complex structure $\mathcal{J}$, and suppose that $(M, \mathcal{J})$ satisfies the $\bar{\partial}\partial$-lemma, then the induced G action on $H^*_{\bar{\partial}}(M)$ and $H^*(\Omega(M), \partial)$ is trivial.*

Proof. We first claim that the induced action of G on $H_H(M)$ is trivial. Since G is connected, it suffices to show that all operators L_{ξ_M} act trivially on $H(M, H)$, where ξ_M denotes the vector field on M induced by $\xi \in \mathfrak{g}$. However, since $\iota_{\xi_M}(H \wedge \beta) = -H \wedge (\iota_{\xi_M}\beta)$ for any form β, we have

$$L_{\xi_M} = d\iota_{\xi_M} + \iota_{\xi_M}d = (d - H\wedge)\iota_{\xi_M} + \iota_{\xi_M}(d - H\wedge) = d_H\iota_{\xi_M} + \iota_{\xi_M}d_H.$$

That is to say that L_{ξ_M} is chain homotopic to zero in $H(M, H)$ and our claim is established.

Let α be a representative of an element $[\alpha]$ of $H_{\bar{\partial}}(M)$. By Proposition 2.6 we may well assume that α is both ∂ and $\bar{\partial}$ closed. In particular, this implies that α is d_H-closed. Let g be an element of G. Since the induced Lie group action on the H-twisted de Rham cohomology is trivial, we have

$$g^*\alpha - \alpha = d\gamma \tag{4.2}$$

for some $\gamma \in \Omega(M)$. Without the loss of generality we may assume that γ has only U^{k-1} component γ^{k-1} and U^{k+1} component γ^{k+1}. By comparing the components of the both sides of (4.2) we get

$$g^*\alpha - \alpha = \bar{\partial}\gamma^{k-1} + \partial\gamma^{k+1}, \qquad \partial\gamma^{k-1} = 0, \qquad \bar{\partial}\gamma^{k+1} = 0.$$

Note that $\partial\gamma^{k+1}$ is both ∂ exact and $\bar{\partial}$ closed. By the $\bar{\partial}\partial$-lemma $\partial\gamma^{k+1} = \bar{\partial}\partial\eta$ for some $\eta \in U^k$. Therefore $g^*\alpha - \alpha = \bar{\partial}(\gamma^{k-1} + \partial\eta)$. This shows that the induced G action on $H_{\bar{\partial}}(M)$ is trivial. A similar argument shows that the induced G action on $H(\Omega(M), \partial)$ is trivial. □

It is important to notice that ∂ is not a derivation. But we have the following Leibniz rule.

Lemma 4.8. *Let f be the generalized moment map and let $\eta \in \Omega^1(M, \mathfrak{g}^*)$ be the associated moment 1-form. For any $\xi \in \mathfrak{g}$, define $\mathcal{A}(\xi) = -\xi + \sqrt{-1}(\mathrm{d}f^\xi + \sqrt{-1}\eta^\xi)$ as before. Then for any $\alpha \in \Omega(M)$, we have*

$$\partial(f^\xi\alpha) = -\frac{\sqrt{-1}}{2}\mathcal{A}(\xi)\cdot\alpha + f^\xi\partial\alpha.$$

Proof. Without the loss of generality we may assume that $\alpha \in U^k$. First we note that

$$d(f^\xi\alpha) = \mathrm{d}f^\xi \wedge \alpha + f^\xi\mathrm{d}\alpha.$$

It is easily seen that $\mathrm{d}f^{\xi}=-\frac{\sqrt{-1}}{2}\mathcal{A}(\xi)+\frac{\sqrt{-1}}{2}\overline{\mathcal{A}(\xi)}$ with $-\frac{\sqrt{-1}}{2}\mathcal{A}(\xi)\in C^{\infty}(L)$ and $\frac{\sqrt{-1}}{2}\overline{\mathcal{A}(\xi)}\in C^{\infty}(\overline{L})$. Thus

$$d(f^{\xi}\alpha)=\left(-\frac{\sqrt{-1}}{2}\mathcal{A}(\xi)+\frac{\sqrt{-1}}{2}\overline{\mathcal{A}(\xi)}\right)\cdot\alpha+f^{\xi}\mathrm{d}\alpha.$$

Now compare the U^{k-1} component of the both side of the above equality, we get that

$$\partial(f^{\xi}\alpha)=-\frac{\sqrt{-1}}{2}\mathcal{A}(\xi)\cdot\alpha+f^{\xi}\partial\alpha. \quad \square$$

We note that as an immediate consequence of Lemmas 4.7 and 4.8, the homology complex $H(U_G,\partial)$ is a double complex with trivial differentials if the manifold M satisfies the $\bar{\partial}\partial$-lemma.

Lemma 4.9. *If $(M,\mathcal{J})$ satisfies the $\bar{\partial}\partial$-lemma, then both differentials $\bar{\partial}$ and $\mathcal{A}$ on $H(U_G,\partial)$ are zero.*

Proof. First we observe that the (ordinary) $\bar{\partial}\partial$-lemma holds for equivariant forms as well as for ordinary forms. The reason is that ∂ and $\bar{\partial}$ acts on U_G as $1\otimes\partial$ and $1\otimes\bar{\partial}$ respectively and the both operators are G equivariant. Now suppose $\alpha\in U_G$ satisfies that $\partial\alpha=0$. Then $\bar{\partial}\alpha=\bar{\partial}\partial\beta=-\partial\bar{\partial}\beta$ for some $\beta\in U_G$. Hence the differential on $H(U_G,\partial)$ induced by $\bar{\partial}$ is zero.

To prove the other differential is zero we have to be more careful to pick a representative of an element of $H(U_G,\partial)$. By Lemma 4.7 the induced G action on $H(\Omega(M),\partial)$ is trivial. This implies that

$$H(U_G,\partial)=\left(S\mathfrak{g}^*\otimes H(\Omega(M),\partial)\right)^G=(S\mathfrak{g}^*)^G\otimes H(\Omega(M),\partial). \tag{4.3}$$

Choose a basis ξ_i of $\mathfrak{g}$. Let x_i be the dual basis of $\mathfrak{g}^*$ and let f_i be a basis of the vector space $(S\mathfrak{g}^*)^G$ of invariant polynomials. It follows from (4.3) that an element of $H(U_G,\partial)$ can be represented by an $\alpha\in U_G$ with $\partial\alpha=0$ of the form $\alpha=\sum_i f_i\otimes\alpha_i$ for unique $\alpha_i\in\Omega^G(M)$. It follows that $\partial\alpha_i=0$ for all i. And so by Lemma 4.8 $\mathcal{A}(\xi_j)\alpha_i=\partial\beta_{ij}$. Hence $\mathcal{A}\alpha=\sum_{i,j}x_jf_i\otimes\partial\beta_{ij}=\partial(\sum_{i,j}x_jf_i\otimes\beta_{ij})$. Since the operator $\mathcal{A}$ and ∂ are equivariant, after averaging over G we get $\mathcal{A}\alpha=\partial\beta$ with $\beta\in U_G$, i.e., the differential on $H(U_G,\partial)$ induced by $\mathcal{A}$ is trivial. $\square$

Let E be the spectral sequence of U_G relative to the filtration associated to the horizontal grading and E_∂ that of $U_{G,\partial}$. The first terms are

$$\begin{aligned}E_1&=\ker\bar{\partial}/\mathrm{im}\,\bar{\partial}=\left(S\mathfrak{g}^*\otimes H_{\bar{\partial}}(M)\right)^G=(S\mathfrak{g}^*)^G\otimes H_{\bar{\partial}}(M),\\(E_\partial)_1&=(\ker\bar{\partial}\cap\ker\partial)/(\mathrm{im}\,\bar{\partial}\cap\ker\partial)=(S\mathfrak{g}^*\otimes H(\Omega_\partial(M),\bar{\partial}))^G=(S\mathfrak{g}^*)^G\otimes H_{\bar{\partial}}(M).\end{aligned}\tag{4.4}$$

Here we used the isomorphism $H(\Omega_\partial(M),\bar{\partial})\cong H_{\bar{\partial}}(M)$ of Proposition 2.6 and the connectedness of G. By Lemma 4.9 $H(U_G,\partial)$ is a double complex with trivial differentials, so its spectral sequence is constant with trivial differential at each stage. The two morphisms (4.1) induce morphisms of spectral sequences

$$E\leftarrow E_\partial\rightarrow H(U_G,\partial).$$

It follows from (4.3) and (4.4) that these two morphisms are isomorphisms at the first stage and hence are isomorphisms at every stage. So they induce isomorphisms on the total cohomology. In fact, since the spectral sequence for $H(U_G,\partial)$ is constant, so are spectral sequences E and E_∂. This proves the following result, where $H_{G,\partial}(M)$ denotes the total cohomology of $U_{G,\partial}$.

Theorem 4.10 (*Equivariant Formality I*). *Assume that the generalized complex manifold M satisfies the $\bar{\partial}\partial$-lemma. Then the spectral sequences E and E_∂ degenerate at the first terms. And the morphisms (4.1) induce isomorphisms of $(S\mathfrak{g}^*)^G$-modules*

$$H(U_G,\bar{\partial}_G)\leftarrow H_{G,\partial}(M)\rightarrow(S\mathfrak{g}^*)^G\otimes H_\partial(M).$$

The following corollary is an immediate consequence of Theorem 4.10 and Proposition 2.6.

Corollary 4.11. *Assume that the generalized complex manifold M satisfies the $\bar\partial\partial$-lemma. As $(S\mathfrak{g}^*)^G$-modules*

$$H(U_G, \bar\partial_G) \cong (S\mathfrak{g}^*)^G \otimes H_H(M).$$

To prove the $\bar\partial_G\partial$-lemma we need the following useful technical lemma. For a proof, we refer to [23].

Lemma 4.12 (*[23]Dδ-Lemma*). *Let (K^{**}, d, d') be a double complex which is bounded in the following sense: for each n, there are only finitely many non-zero components in the direct sum $K^n = \bigoplus_{i+j=n} K^{i,j}$. Here d is the degree 1 vertical differential and d' the degree 1 horizontal differential. Assume that there is a degree −1 vertical differential δ which anti-commutes with both d and d', i.e., $d\delta = -\delta d$, $d'\delta = -\delta d'$. Let (K^*, D) be the associated total complex, where $D = d + d'$. And assume that the double complex (K^{**}, d, d') satisfies:*

(a) $\operatorname{im} d \cap \ker \delta = \ker d \cap \operatorname{im} \delta = \operatorname{im} d\delta$;
(b) *The spectral sequence associated to the row filtration degenerates at the E_1 stage.*

Then we have $\operatorname{im} D \cap \ker \delta = \operatorname{im} D\delta$.

We are ready to state the main result of this section.

Theorem 4.13. *Consider the Hamiltonian action of a connected compact Lie group on an H-twisted generalized complex manifold $(M, \mathcal{J})$. If $(M, \mathcal{J})$ satisfies the $\bar\partial\partial$-lemma, then*

$$\operatorname{im} \bar\partial_G \cap \ker \partial = \ker \bar\partial_G \cap \operatorname{im} \partial = \operatorname{im} \bar\partial_G \partial.$$

Proof. As a direct consequence of Theorem 4.10 and Lemma 4.12, we have $\operatorname{im} \bar\partial_G \cap \ker \partial = \operatorname{im} \bar\partial_G\partial$. The second half of the $\bar\partial_G\partial$-lemma follows from the first: assume that $\bar\partial_G\alpha = 0$ and α is ∂ exact. Then the cohomology class of α in $H(U_G, \partial)$ is zero, so by Theorem 4.10 the cohomology class of α in $H_{G,\partial}(M)$ is zero, i.e., α is $\bar\partial_G$ exact. Hence $\alpha = \bar\partial_G\partial\beta$ for some β. □

As an easy consequence, the $\bar\partial_G\partial$-lemma holds for compact H-generalized Kähler manifolds. To state this result more precisely, let us first recall the definition of Hamiltonian actions on twisted generalized Kähler manifolds.

Definition 4.14 (*[25]*). Let the compact Lie group G with Lie algebra $\mathfrak{g}$ act on a manifold M. A *generalized moment map* for an invariant H-twisted generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ is a generalized moment map for the generalized complex structure $\mathcal{J}_1$. If such a generalized moment map exists, the action on the H-twisted generalized Kähler manifold $(M, \mathcal{J}_1, \mathcal{J}_2)$ is said to be Hamiltonian.

Corollary 4.15. *Assume that the action of the compact Lie group G on an H-twisted generalized Kähler manifold $(M, \mathcal{J}_1, \mathcal{J}_2)$ is Hamiltonian. Then the $\bar\partial_G\partial$-lemma holds for the generalized complex manifold $(M, \mathcal{J}_1)$.*

Proof. By Theorem 2.7, the $\bar\partial\partial$-lemma holds on M with respect to both $\mathcal{J}_1$ and $\mathcal{J}_2$. By definition, the action of G is Hamiltonian on the generalized complex manifold $(M, \mathcal{J}_1)$. Now the corollary follows easily from Theorem 4.13. □

To conclude this section let us present an application of the $\bar\partial_G\partial$-lemma which says that the generalized equivariant cohomology is canonically isomorphic to $(S\mathfrak{g}^*)^G \otimes H_H(M)$ provided the manifold M satisfies the $\bar\partial\partial$-lemma.

Observe that the inclusion map

$$(U_{G,\partial}, \bar\partial_G) \hookrightarrow (\Omega_G, D_G)$$

is actually a chain map with respect to the differentials $\bar\partial_G$ and D_G since $\bar\partial_G\alpha = D_G\alpha$ for any $\alpha \in U_{G,\partial} = U_G \cap \ker \partial$. So it induces a map

$$H_{G,\partial}(M) \to H(\Omega_G, D_G). \tag{4.5}$$

Suppose α is a representative of a cohomology class in $H_{G,\partial}(M)$ and $\alpha = D_G\beta$ for some $\beta \in \Omega_G$. Then $\alpha - \partial\beta = \bar\partial_G\beta$ is both ∂ closed and $\bar\partial_G$ exact. So by the $\bar\partial_G\partial$-lemma $\alpha - \partial\beta = \bar\partial_G\partial\gamma$ for some $\gamma \in U_G$. Thus $\alpha = \partial(\beta - \bar\partial_G\gamma)$ is both ∂-exact and $\bar\partial_G$ closed. Applying the $\bar\partial_G\partial$-lemma again, we conclude that $\alpha = \bar\partial_G\partial\eta$ for some $\eta \in U_G$. This shows that α represents a trivial cohomology class in $H_{G,\partial}(M)$ and the map (4.5) is injective.

Now suppose α is a representative of a cohomology class in $H(\Omega_G, D_G)$. Then since $\bar{\partial}_G\partial\alpha = -\partial D_G\alpha = 0$, $\partial\alpha$ is both ∂ exact and $\bar{\partial}_G$ closed. So $\partial\alpha = \bar{\partial}_G\partial\beta$ for some $\beta \in U_G$. It follows that $\partial(\alpha + \bar{\partial}_G\beta) = \partial(\alpha + D_G\beta) = 0$. Since $D_G\alpha = \bar{\partial}_G\alpha + \partial\alpha = 0$, $\bar{\partial}_G(\alpha + D_G\beta) = -\partial\alpha + \bar{\partial}_G\partial\beta = 0$. This shows clearly that the cohomology class of α in $H(\Omega_G, D_G)$ is the image of the cohomology class of $\alpha + D_G\beta$ in $H_{G,\partial}(M)$. Hence the map (4.5) is surjective. The above discussion, together with Corollary 4.11, leads to the following theorem.

Theorem 4.16 (*Equivariant Formality II*). *Assume the generalized complex manifold M satisfies the $\bar{\partial}\partial$-lemma. Then*

$$H(\Omega_G, D_G) \cong (S\mathfrak{g}^*)^G \otimes H_H(M).$$

5. Torus actions on generalized Kähler manifolds

Assume that $(M, \mathcal{J})$ is a generalized complex manifold which satisfies the $\bar{\partial}\partial$-lemma. By Proposition 2.6 we have that $H(M) = \bigoplus_k H^k_{\bar{\partial}}(M)$. Therefore we have the following decomposition of $H_G(M)$.

Proposition 5.1. *Assume the connected compact Lie group G action on M is equivariantly formal and assume M satisfies the $\bar{\partial}\partial$-lemma. Then there is a canonical isomorphism of $(S\mathfrak{g}^*)^G$-modules*

$$H_G(M) = \bigoplus_k (S\mathfrak{g}^*)^G \otimes H^k_{\bar{\partial}}(M). \tag{5.1}$$

Remark 5.2. It is obvious that the above canonical isomorphism depends only on the invariant metric that we choose. When the generalized complex structure $\mathcal{J}$ is induced by a complex structure I in an G-invariant Kähler pair (ω, I), it is easy to recover from (5.1) the equivariant Dolbeault decomposition as treated by Teleman [29] and Lillywhite [26]. Indeed, assuming the group action is equivariantly formal, using the $\bar{\partial}\partial$-lemma for compact Kähler manifolds it is not difficult to show directly that, for any $\bar{\partial}$-closed differential form α, any holomorphic vector Z generated by the group action and any $p > 0$, there exist differential forms $\alpha_1, \ldots, \alpha_p$ so that

$$\iota_Z\alpha = \bar{\partial}\alpha_1, \ldots, \iota_Z\alpha_{p-1} = \bar{\partial}\alpha_p.$$

Then one can apply Allday's argument in [1] to the invariant Kähler metric and prove that the right hand side of (5.1) is canonically isomorphic to the equivariant Dolbeault cohomology. This approach will give us a new Hodge theoretic proof of the usual equivariant Dolbeault decomposition without using any Morse theory.

Henceforth we will assume that $G = T$ is a connected compact torus and that the T action preserves the generalized complex structure $\mathcal{J}$. Our next observation is that the fixed point submanifold of the T action is a generalized complex submanifold in the sense specified in [3]. However, for the convenience of the reader, let us first review the notion of generalized complex submanifolds [3].

Let W be a submanifold of the generalized complex manifold $(M, \mathcal{J})$ and let L be the $\sqrt{-1}$ eigenbundle of $\mathcal{J}$. Then at each point $x \in N$ define

$$L_{W,x} = \left\{X + (\xi\,|_{T_{\mathbb{C}}W}) : X + \xi \in L \cap \left(T_{\mathbb{C},x}W \;\oplus\; T^*_{\mathbb{C},x}M\right)\right\}.$$

This actually defines a Dirac structure on W, i.e., a maximal isotropic distribution $L_W \subset T_{\mathbb{C}}W \oplus T^*_{\mathbb{C}}W$ whose sections are closed under the Courant bracket. If L_W is such that $L_W \cap \overline{L_W} = 0$, then W is said to be a *generalized complex submanifold* of M. It is clear from the definition that if W is a generalized complex submanifold of M, then there is a unique generalized complex structure $\mathcal{J}_W$ on W whose $\sqrt{-1}$ eigenbundle is exactly L_W. Moreover, [3] gives a simple condition, the "split condition", to ensure the submanifold W of M is a generalized complex submanifold. Specifically, W is said to be *split* if there exists a smooth subbundle N of $TM\,|_W$ such that $TM\,|_W = TW \oplus N$ and such that $TW \oplus \mathrm{Ann}(N)$ is invariant under the generalized complex structure $\mathcal{J}$. Here $\mathrm{Ann}(N) \subset T^*M\,|_W$ denotes the annihilator of N.

Proposition 5.3 (*[3]*). *Let $(M, \mathcal{J})$ be a generalized complex manifold and let W be a split submanifold. Then W is a generalized complex submanifold of M. Moreover, let $\psi : TW \oplus \mathrm{Ann}(N) \to TW \oplus T^*W$ be the natural isomorphism, then the induced generalized complex structure $\mathcal{J}_W$ on W has the form $\mathcal{J}_W = \psi \circ \mathcal{J} \circ \psi^{-1}$.*

Recall that the fixed point submanifold of a symplectic torus action on a symplectic manifold is a symplectic submanifold. The following lemma is a generalization of this well-known fact. We note that in the case of Z_2 actions on generalized complex manifolds the similar result has been obtained by Barton and Stiénon [2] using different methods.

Lemma 5.4. *Suppose that the torus T acts on the generalized complex manifold $(M, \mathcal{J})$ preserving the generalized complex structure $\mathcal{J}$. And suppose that Z is a connected component of the fixed point submanifold. Then Z is a split submanifold and so is a generalized complex submanifold of M.*

Proof. Let $x \in Z$ be a fixed point of the torus action. Then the action of the torus T on M induces a T-module structure on T_xM and a dual T-module structure on T_x^*M. Let $\{\vartheta_1, \vartheta_2, \dots, \vartheta_k\}$ be the set of all distinct weights of the T-module T_xM, where $\vartheta_1 \equiv 1$. Then $\{\vartheta_1^{-1}, \vartheta_2^{-1}, \dots, \vartheta_k^{-1}\}$ is the set of all distinct weights of the dual T-module T_x^*M. Let $V_i \subset T_xM$ be the weight space corresponding to ϑ_i and $V_i^* \subset T_x^*M$ the weight space corresponding to ϑ_i^{-1}, $1 \le i \le k$, and let $N_x = \bigoplus_{i=2}^k V_i$. Then it is clear $V_1 = T_xZ$. Moreover, we claim that $\mathrm{Ann}(N_x)$, the annihilator of N_x, coincides with V_1^*. Indeed, for any $u^* \in V_1^*$, $w \in V_i$, $i \ge 2$, we have

$$\langle w, u^* \rangle = \langle tw, tu^* \rangle = \langle \vartheta_i(t)w, u^* \rangle = \vartheta_i(t)\langle w, u^* \rangle,$$

where t is an arbitrarily chosen element in T. It follows $\langle u^*, w \rangle = 0$ and so $V_1^* \subset \mathrm{Ann}(N_x)$. A dimension count shows that we actually have $V_1^* = \mathrm{Ann}(N_x)$. Since the torus action preserves the generalized complex structure $\mathcal{J}$, $V_1 \oplus V_1^* = T_xZ \oplus \mathrm{Ann}(N_x)$ is invariant under $\mathcal{J}$. Put $N = \bigcup_x N_x$. Then N is a smooth subbundle of $TM|_Z$ such that $TM|_Z = TZ \oplus N$ and such that $TZ \oplus \mathrm{Ann}(N)$ is invariant under $\mathcal{J}$. This completes the proof that Z is a split submanifold. □

Proposition 5.5. *Suppose that the torus T action preserves the generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$. And suppose Z is a connected component of the fixed point submanifold of the torus action. Then Z is split with respect to both $\mathcal{J}_1$ and $\mathcal{J}_2$. Furthermore, the induced generalized complex structures $\mathcal{J}_{1,Z}$ and $\mathcal{J}_{2,Z}$ defines a generalized Kähler pair on Z.*

Proof. The first assertion is an immediate consequence of Lemma 5.4. The second assertion follows from the description of the induced generalized complex structures on a split submanifold given in Proposition 5.3. □

Example 5.6. As explained in [3], if $\mathcal{J}_\omega$ is a generalized complex structure induced by a symplectic structure ω on M, then Z is a generalized complex submanifold of $(M, \mathcal{J}_\omega)$ if and only if Z is a symplectic submanifold of (M, ω). By Example 2.5

$$\Gamma\left(U^k(M)\right) = \{e^{i\omega}e^{\frac{\wedge}{2i}}\alpha : \alpha \in \Omega^{n+k}(M)\}, \qquad \Gamma\left(U^k(Z)\right) = \{e^{i\omega_0}e^{\frac{\wedge_0}{2i}}\alpha : \alpha \in \Omega^{n+k}(Z)\},$$

where ω_0 is the restriction of ω to Z and $\wedge_0$ is the Poisson bi-vector on Z associated to the symplectic structure ω_0. It is then not hard to see that $\alpha \in \Gamma(U^k(M))$ does not necessarily imply that $(\alpha|_Z) \in \Gamma(U^k(Z))$. As a result, we see $\bar\partial\alpha = 0$ on M does not necessarily imply that $\bar\partial_Z(\alpha|_Z) = 0$, where $\bar\partial_Z$ is $\bar\partial$ operator associated to the generalized complex structure on Z induced by the symplectic structure ω_0.

Example 5.7.
- Let V be a real vector space with a complex structure I, and let W be a subspace of V which is invariant under I. It is clear that W inherits a complex structure I_W from (V, I). Denote by $\mathcal{J}$ and $\mathcal{J}_W$ the generalized complex induced by I and I_W respectively. Let $U^k(V)$ and $U^k(W)$ be the $-k\sqrt{-1}$ eigenspace of the Clifford actions of $\mathcal{J}$ and $\mathcal{J}_W$ respectively. Then by Example 2.4

$$U^k(V) = \bigoplus_{p-q=k} \wedge^{p,q} V^*, \qquad U^k(W) = \bigoplus_{p-q=k} \wedge^{p,q} W^*.$$

  In particular, if $\alpha \in U^k(V)$, then $(\alpha|_W) \in U^k(W)$.
- Suppose that $\mathcal{J}$ is a generalized complex structure on M induced by a complex structure I. Then as shown in [3] a submanifold Z is a generalized complex submanifold of $(M, \mathcal{J})$ if and only if S is a complex submanifold of (M, I).

Now suppose that Z is a generalized complex submanifold of $(M, \mathcal{J})$ with the induced generalized complex structure $\mathcal{J}_Z$. By the preceding discussion, if $\alpha \in \Gamma(U^k(M))$, then $(\alpha\,|_Z) \in \Gamma(U^k(Z))$. So if $\alpha \in \Gamma(U^k(M))$ such that $\bar\partial\alpha = 0$, then $\bar\partial_Z(\alpha\,|_Z) = 0$, where $\bar\partial_Z$ is the $\bar\partial$-operator associated to the generalized complex structure $\mathcal{J}_Z$.

The same argument actually gives us the following slightly more general result.

Lemma 5.8. *Let $(M, \mathcal{J})$ be a generalized complex manifold and let Z be a generalized complex submanifold with the generalized complex structure $\mathcal{J}_Z$. Suppose that for each $x \in Z$, $\mathcal{J}_x$ is induced by a complex structure on the tangent space T_xM. As a result, the generalized complex structure $\mathcal{J}_Z$ on Z is induced by a complex structure on Z. Furthermore if $\alpha \in \Gamma(U^k(M))$, then $(\alpha\,|_Z) \in \Gamma(U^k(Z))$. In particular, this implies that if $\bar\partial\alpha = 0$, then $\bar\partial_Z(\alpha\,|_Z) = 0$, where $\bar\partial_Z$ is the $\bar\partial$-operator associated to the generalized complex structure $\mathcal{J}_Z$.*

Theorem 5.9. *Consider the action of a torus T on a generalized Kähler manifold M which preserves the generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$. Assume that the action is equivariantly formal. Let Z be the fixed point submanifold. And assume that for any $x \in Z$, $\mathcal{J}_{2,x}$, the restriction of $\mathcal{J}_2$ to T_xM, is induced by a complex structure on T_xM. Then*

$$H^i_{\bar\partial_2}(M) = 0 \quad \textit{if } |i| > \dim Z,$$

where $\bar\partial_2$ is the $\bar\partial$-operator associated to the generalized complex structure $\mathcal{J}_2$.

Proof. Let us denote by S the polynomial ring over the Lie algebra $\mathfrak{t}$ of T. By Proposition 5.1 there is a canonical isomorphism $H_T(M) \cong \bigoplus_k S \otimes H^k_{\bar\partial}(M)$. By Proposition 5.5 the fixed point submanifold Z is a compact generalized Kähler manifold and therefore satisfies the $\bar\partial\partial$-lemma with respect to both induced generalized complex structures. This implies that $H_T(Z) \cong \bigoplus_k S \otimes H^k_{\bar\partial}(M)$. A straightforward check shows that there is a commutative diagram

$$\begin{array}{ccc} H_T(M) & \xrightarrow{\text{isomorphism}} & \bigoplus_k S \otimes H^k_{\bar\partial}(M) \\ {\scriptstyle\text{injection}}\downarrow & & \downarrow \\ H_T(Z) & \xrightarrow{\text{isomorphism}} & \bigoplus_k S \otimes H^k_{\bar\partial}(Z), \end{array}$$

where the right vertical map is well defined because of Lemma 5.8. Observe that the direct summand $S \otimes H^k_{\bar\partial}(M)$ is mapped into $S \otimes H^k_{\bar\partial}(Z)$; moreover, since the above diagram is commutative, the map $S \otimes H^k_{\bar\partial}(M) \to S \otimes H^k_{\bar\partial}(Z)$ is injective. By Lemma 5.8 the generalized complex structure on Z is induced by a complex structure and so $\Gamma(U^k(Z)) = 0$ if $|i| > \dim Z$ by dimension consideration. Therefore for any $|i| > \dim Z$, $H^k_{\bar\partial}(Z) = 0$ and so $H^k_{\bar\partial}(M) = 0$. $\square$

When the invariant generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ is induced by an invariant Kähler structure (ω, I), it is easy to recover from Theorem 5.9 the following result of Carrell and Lieberman [6,7].

Corollary 5.10. *Suppose that M is a compact Kähler manifold with an equivariantly formal torus action which preserves the Kähler structure. And suppose that the fixed point submanifold Z of the torus action is non-empty. Assume that $|p - q| > \dim Z$. Then $H^{p,q}_{\bar\partial}(M) = 0$.*

Proof. Since the fixed point of the torus action is non-empty, by a well-known result of Frankel [9] the torus action is Hamiltonian and so is equivariantly formal by Kirwan–Ginzburg equivariant formality theorem. Let $(\mathcal{J}_1, \mathcal{J}_2)$ be the generalized Kähler structure induced by the Kähler structure (ω, I) on M. Then it is easy to check that the assumptions in Theorem 5.9 are all satisfied. Thus $H^k_{\bar\partial_2}(M) = 0$ if $|k| > \dim Z$. Let U^k be the $-k\sqrt{-1}$ eigenbundle of the generalized complex structure $\mathcal{J}_2$. It then follows from Example 2.4 that $U^i = \bigoplus_{q-p=i}(\wedge^{p,q} T^*_{\mathbb{C}}M)$ and that the $\bar\partial$ operator associated to the generalized complex structure $\mathcal{J}_2$ coincides with the usual $\bar\partial$ associated to the complex structure I. This finishes the proof. $\square$

6. Calculation of generalized Hodge numbers

The generalized Hodge theory for compact generalized Kähler manifolds has been established by Gualtieri [11]. Let us recall some salient points of this theory and refer to [11] for details.

Let $(M, \mathcal{J}_1, \mathcal{J}_2)$ be a compact generalized Kähler manifold of dimension $2n$. Then $-\mathcal{J}_1\mathcal{J}_2$, regarded as a positive definite metric on $TM \oplus T^*M$, induces a Hermitian inner product, the *Born–Infeld* inner product, on the space of differential forms.

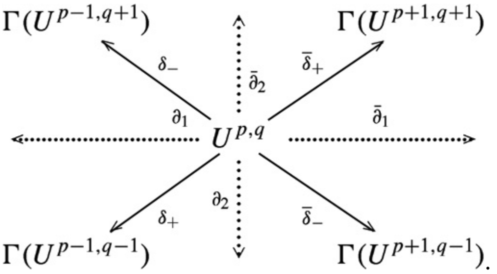
Let $\Gamma(U^k)$ be the $-\sqrt{-1}$ eigenspace of the Clifford action of $\mathcal{J}_1$ on the space of differential forms. The commuting endomorphism $\mathcal{J}_2$ further decomposes U^k as

$$\Gamma(U^k) = \Gamma(U^{k,|k|-n}) \oplus \Gamma(U^{k,|k|-n+2}) \oplus \cdots \oplus \Gamma(U^{k,n-|k|}).$$

Thus there is a (p, q) decomposition of the differential forms. Furthermore, this decomposition is orthogonal with respect to the *Born–Infeld* metric, and gives rise to the following splitting of the exterior derivative:

$$d = \delta_+ + \delta_- + \overline{\delta}_+ + \overline{\delta}_-.$$

Here the differential operators act as follows:



Let d^*, $\bar\partial^*$, ∂^*, $\delta_\pm^*$ and $\overline{\delta}_\pm^*$ be the adjoint operators of d, $\bar\partial$, ∂, $\delta_\pm$ and $\overline{\delta}_\pm$ with respect to the *Born–Infeld* inner product respectively. Then $d + d^*$, $\bar\partial_{1/2} + \bar\partial_{1/2}^*$, $\partial_{1/2} + \partial_{1/2}$, $\delta_\pm + \delta_\pm^*$, $\overline{\delta}_\pm + \overline{\delta}_\pm^*$ are all elliptic operators; moreover, we have

$$\triangle_d = 2\,\triangle_{\partial_{1/2}} = 2\,\triangle_{\bar\partial_{1/2}} = 4\,\triangle_{\delta_\pm} = 4\,\triangle_{\overline{\delta}_\pm},$$

where $\triangle_d$, $\triangle_{\partial_{1/2}}$, $\triangle_{\bar\partial_{1/2}}$, $\triangle_{\delta_\pm}$, $\triangle_{\overline{\delta}_\pm}$ are the Laplacians of d, $\partial_{1/2}$, $\bar\partial_{1/2}$, $\bar\partial_{1/2}$, $\delta_\pm$ and $\bar\partial_\pm$ respectively. It then follows from the standard Hodge theory for elliptic operators

Theorem 6.1 (*[11]*). *The cohomology of a compact $2n$-dimensional generalized Kähler manifold carries a Hodge decomposition*

$$H^*(M, \mathbb{C}) = \bigoplus_{\substack{|p+q| \le n \\ p+q \equiv 2 (\mathrm{mod}\ 2)}} \mathcal{H}^{p,q},$$

where $\mathcal{H}^{p,q}$ are $\triangle_d$ harmonic forms in $\Gamma(U^{p,q})$.

The *generalized Hodge number* $h^{p,q}$ of a generalized Kähler manifold $(M, \mathcal{J}_1, \mathcal{J}_2)$ is then defined to be the complex dimension of $\mathcal{H}^{p,q}$. The case of interest is when $(\mathcal{J}_1, \mathcal{J}_2)$ is not the B-transform of a genuine Kähler structure. (cf. Example 2.8.) In this paper, we refer to such generalized Kähler structures as *non-trivial*. Note that [25] proposes a general method of constructing non-trivial explicit examples of generalized Kähler structures as the generalized Kähler quotient of the vector space $\mathbb{C}^n$. In the rest of this section, we are going to compute the generalized Hodge number for two of these examples. But let us first recall how to construct non-trivial examples of generalized Kähler manifolds as generalized Kähler quotient.

Let $\mathcal{J}$ be a generalized complex structure on a vector space $V = \mathbb{C}^n$. Let $L \subset V_{\mathbb{C}} \oplus V_{\mathbb{C}}^*$ be the $\sqrt{-1}$ eigenspace of $\mathcal{J}$. Since L is maximal isotropic and $L \cap \overline{L} = \{0\}$, we can (and will) use the metric to identify L^* with $\overline{L}$.

Given $\epsilon \in \wedge^2 L^*$, define $L_\epsilon = \{Y + \iota_Y \epsilon \mid Y \in L\}$. Then L_ϵ is maximal isotropic, and $L_\epsilon \cap \overline{L_\epsilon} = \{0\}$ if and only if the endomorphism

$$A_\epsilon = \begin{pmatrix} 1 & \bar{\epsilon} \\ \epsilon & 1 \end{pmatrix} : L \oplus \overline{L} \to L \oplus \overline{L} \tag{6.1}$$

is invertible. If it is invertible, there exists a unique generalized complex structure $\mathcal{J}_\epsilon$ on V whose $\sqrt{-1}$ eigenspace is L_ϵ. Note that A_ϵ is always invertible for ϵ sufficiently small.

Now let $(\mathcal{J}_\omega, \mathcal{J}_I)$ be the generalized Kähler structure on $V = \mathbb{C}^n$ which is induced by the standard genuine Kähler structure (ω, I). Let L_1 and L_2 denote the $\sqrt{-1}$ eigenspaces of $\mathcal{J}_\omega$ and $\mathcal{J}_I$ respectively. Then $L_1 = (L_1 \cap L_2) \oplus (L_1 \cap \overline{L_2})$ and $L_2 = (L_1 \cap L_2) \oplus (\overline{L_1} \cap L_2)$. Thus $\epsilon \in C^\infty(\wedge^2 \overline{L_2})$ fixes $\mathcal{J}_\omega$ if and only if ϵ takes $L_1 \cap L_2$ to $L_1 \cap \overline{L_2}$, i.e., if and only if ϵ is an element of $C^\infty((\overline{L_1} \cap \overline{L_2}) \otimes (L_1 \cap \overline{L_2}))$.

Henceforth we will assume that $\epsilon \in C^\infty((\overline{L_1} \cap \overline{L_2}) \otimes (L_1 \cap \overline{L_2}))$. So it will fix $\mathcal{J}_\omega$ and deform $\mathcal{J}_I$ to a new generalized almost complex structure $\mathcal{J}_\epsilon$ on a bounded region of $\mathbb{C}^n$. The following lemma gives a simple condition which guarantees that L_ϵ, the $\sqrt{-1}$ eigenbundle of $\mathcal{J}_\epsilon$, is closed under the Courant bracket, and hence that $(\mathcal{J}_\omega, \mathcal{J}_\epsilon)$ is a generalized Kähler structure.

Lemma 6.2. *Assume that there exists a subset $I \subset (1, \ldots, n)$ so that*

$$\epsilon = \sum_{i,j \in I} F_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} + \sum_{i,j \in I} F_{ij}(z) d\overline{z_i} \wedge d\overline{z_j}.$$

If F_{ij} is holomorphic and $\frac{\partial F_{ij}}{\partial z_k} = 0$ for all i, j and $k \in I$, then L_ϵ is closed under the Courant bracket.

Consider the action of a compact connected torus T on $V = \mathbb{C}^n$ which preserves the generalized Kähler structure $(\mathcal{J}_\omega, \mathcal{J}_\epsilon)$. A *generalized moment map* for $(\mathcal{J}_\omega, \mathcal{J}_\epsilon)$ is the generalized moment map for the generalized complex structure $\mathcal{J}_\omega$ which coincides with the usual moment map for the symplectic structure ω in this context. Let $\mathfrak{t}$ be the Lie algebra of the torus T and $f : M \to \mathfrak{t}^*$ be the generalized moment map for the torus action. If $a \in \mathfrak{t}^*$ so that T acts freely on $f^{-1}(a)$, then $M_a = f^{-1}(a)/S^1$ is defined to be the *generalized Kähler quotient* of the torus action at the level a. There is a naturally defined generalized Kähler structure $(\widetilde{\mathcal{J}_1}, \widetilde{\mathcal{J}_2})$ on M_a; moreover, for any $m \in f^{-1}(a)$ we have

$$\begin{aligned} \text{type}(\widetilde{\mathcal{J}_\omega})_{[m]} &= \text{type}(\mathcal{J}_\omega)_m = 0, \\ \text{type}(\widetilde{\mathcal{J}_\epsilon})_{[m]} &= \text{type}(\mathcal{J}_\epsilon)_m - \dim(T) + 2\dim(\mathfrak{t}_M \cap \pi(L_\epsilon))_m, \end{aligned} \tag{6.2}$$

where $\pi : T_{\mathbb{C}}\mathbb{C}^n \oplus T^*_{\mathbb{C}}\mathbb{C}^n \to T_{\mathbb{C}}\mathbb{C}^n$ is the projection map, $\mathfrak{t}_M$ is the distribution of fundamental vector fields generated by the torus action, and L_ϵ is the $\sqrt{-1}$ eigenbundle of the generalized complex structure $\mathcal{J}_\epsilon$. Since a B-transform always preserves the type of a generalized complex structure, $(\widetilde{\mathcal{J}_\omega}, \widetilde{\mathcal{J}_\epsilon})$ is a non-trivial generalized Kähler structure if and only if $\text{type}(\widetilde{\mathcal{J}_\epsilon})_x \neq n - \dim(T)$ at some point x of M_a.

Example 6.3 ($\mathbb{CP}^n$ *for* $n \geq 3$). We now construct a non-trivial generalized Kähler structure on $\mathbb{CP}^n$ for $n \geq 3$ and compute its generalized Hodge number.

Let S^1 act on C^{n+1} via

$$\lambda(z_0, \cdots, z_n) = (\lambda z_0, \ldots, \lambda z_n).$$

Note that this action preserves the Kähler structure (ω, I) with a moment map given by

$$\Phi(z) = \sum_i \frac{1}{2}|z_i|^2;$$

moreover, the S^1 action on $\Phi^{-1}(1)$ is free and the reduced space is exactly $\mathbb{CP}^n$.

Let

$$\epsilon = z_0 z_1 \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial z_3} + z_0 z_1 d\overline{z}_2 \wedge d\overline{z}_3.$$

If necessary, multiplying ϵ by a sufficiently small positive constant, then ϵ will deform $(\mathcal{J}_\omega, \mathcal{J}_I)$ to a new generalized almost Kähler structure on the bounded open set $O = \{\Phi^{-1}(z) < 2\}$ so that type$(\mathcal{J}_\epsilon)_z$ is $n+1$ if $z_0 z_1 = 0$ and is $n-1$ otherwise. Since $z_0 z_1$ is holomorphic and $\frac{\partial z_0 z_1}{\partial z_2} = \frac{\partial z_0 z_1}{\partial z_3} = 0$, by Lemma 6.2 $(\mathcal{J}_\omega, \mathcal{J}_\epsilon)$ is in fact a generalized Kähler structure on O.

Since ϵ is S^1 invariant, $(\mathcal{J}_\omega, \mathcal{J}_\epsilon)$ is also S^1 invariant with a generalized moment map Φ. So there is a naturally defined generalized Kähler structure $(\widetilde{\mathcal{J}}_\omega, \widetilde{\mathcal{J}}_\epsilon)$ on the quotient space $\mathbb{CP}^n = \Phi^{-1}(1)/S^1$. Note that the fundamental vector generated by the above S^1 action on $\mathbb{C}^{n+1}$ is

$$X = \frac{\sqrt{-1}}{2} \sum_i \left(z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} \right),$$

and hence X does not lie in $\pi(L_\epsilon)$ at any point of $\mathbb{C}^{n+1}$, where L_ϵ is the $\sqrt{-1}$ eigenbundle of $\mathcal{J}_\epsilon$. It follows from (6.2) type$(\widetilde{\mathcal{J}}_\omega)_{[z]} = 0$ for all $[z] \in \mathbb{CP}^n$, whereas type$(\widetilde{\mathcal{J}}_\epsilon)_{[z]} = n$ if $z_0 z_1 = 0$, otherwise type$(\widetilde{\mathcal{J}}_\epsilon)_{[z]} = n-2$. So by the preceding discussion the generalized complex structure $(\widetilde{\mathcal{J}}_\omega, \widetilde{\mathcal{J}}_\epsilon)$ on $\mathbb{CP}^n$ is non-trivial.

Next consider the S^1 action on $\mathbb{C}^{n+1}$ defined by

$$\lambda(z_0, z_1, z_2, z_3, \cdots, z_i, \ldots, z_n) = (\lambda z_0, \lambda^5 z_1, \lambda^2 z_2, \lambda^4 z_3, \cdots, \lambda^{2i} z_i, \cdots, \lambda^{2n} z_n).$$

It is easily seen that this action preserves the standard Kähler structure and the deformation ϵ; furthermore it commutes with the standard S^1 action on $\mathbb{C}^{n+1}$ and so descends to an action on $\mathbb{CP}^n$ which preserves the quotient generalized Kähler structure $(\widetilde{\mathcal{J}}_\omega, \widetilde{\mathcal{J}}_\epsilon)$. It is equivariantly formal since the action of any compact Lie group on $\mathbb{CP}^n$ is. Moreover, it is easy to check that this action has $n+1$ isolated fixed points:

$$[(1, 0, \ldots, 0)], [(0, 1, \ldots, 0)], \ldots, [(0, 0, \ldots, 1)].$$

Observe at the tangent space of each fixed point x the generalized Kähler structure $\widetilde{\mathcal{J}}_{\epsilon,x}$ is induced by a complex structure. It then follows from Theorem 5.9 that

$$H^i_{\bar{\partial}_{\widetilde{\mathcal{J}}_\epsilon}}(\mathbb{CP}^n) = 0, \quad \text{if } i \neq 0.$$

Thus the generalized Hodge number $h^{p,q} = 0$ if $q \neq 0$. In addition, we have

$$\begin{aligned} h^{p,0} &= \dim H^p_{\bar{\partial}_{\widetilde{\mathcal{J}}_\omega}}(\mathbb{CP}^n) \quad \text{(By Theorem 6.1.)} \\ &= \dim H^{n+p}(\mathbb{CP}^n) \quad \text{(By Example 2.5.)} \\ &= \begin{cases} 1 & \textit{if} n+p \text{ is even;} \\ 0 & \textit{if} n+p \text{ is odd.} \end{cases} \end{aligned}$$

Example 6.4 ($\mathbb{CP}^n$ *Blown up at one Point for* $n \geq 3$). In this example we construct a non-trivial generalized Kähler structure on the blow up of $\mathbb{CP}^n$ at one point and compute its generalized Hodge number.

Let a two-dimensional torus T with Lie algebra $\mathfrak{t}$ act on $\mathbb{C}^{n+2}$ by

$$(\alpha, \beta)(z_1, \ldots, z_n, z_{n+1}, z_{n+2}) = (\alpha\beta z_1, \ldots, \alpha\beta z_n, \alpha z_{n+1}, \beta^{-1} z_{n+2})$$

with moment map

$$f(z_1, \ldots, z_n, z_{n+1}, z_{n+2}) = (|z_1|^2 + \cdots + |z_n|^2 + |z_{n+1}|^2, |z_1|^2 + \cdots + |z_n|^2 - |z_{n+2}|^2).$$

Then there exists some $\xi \in \mathfrak{t}^*$ so that $M_\xi = f^{-1}(\xi)/T^2$ is equivariantly symplectomorphic to $\mathbb{CP}^n$ blown up at a fixed point.

Define

$$\epsilon = (z_1 z_2 z_{n+2}) \frac{\partial}{\partial z_n} \wedge \frac{\partial}{\partial z_{n+1}} + (z_1 z_2 z_{n+2}) d\bar{z}_n \wedge d\bar{z}_{n+1}.$$

It is clear from construction that ϵ is T invariant. As explained in [25], ϵ deforms the standard Kähler structure $(\mathcal{J}_\omega, \mathcal{J}_I)$ to a T invariant generalized Kähler structure $(\mathcal{J}_\omega, \mathcal{J}_\epsilon)$; moreover, $(\mathcal{J}_\omega, \mathcal{J}_\epsilon)$ descends to a non-trivial generalized Kähler structure $(\widetilde{\mathcal{J}}_\omega, \widetilde{\mathcal{J}}_\epsilon)$ on the reduced space M_ξ, i.e., $\mathbb{CP}^n$ blown up at a fixed point.

Define S^1 action on $\mathbb{C}^{n+2}$ by

$$\alpha(z_1, \ldots, z_n, z_{n+1}, z_{n+2}) = (\alpha^{\lambda_1} z_1, \ldots, \alpha^{\lambda_n} z_n, \alpha^{\lambda_{n+1}} z_{n+1}, \alpha^{\lambda_{n+2}} z_{n+2}),$$

where $\lambda_1, \ldots, \lambda_n, \lambda_{n+1}, \lambda_{n+2}$ are rational numbers so that

(a) $\lambda_1, \ldots, \lambda_n$ are distinct from each other;
(b) $\lambda_1 + \lambda_2 + \lambda_{n+2} = \lambda_n + \lambda_{n+1}$;
(c) $\lambda_i \neq \lambda_{n+1} - \lambda_{n+2}, i = 1, 2, \ldots, n.$

This S^1 action preserves the standard Kähler structure $(\mathcal{J}_\omega, \mathcal{J}_I)$ on $\mathbb{C}^{n+2}$ and the deformation ϵ; furthermore it commutes with the T action and therefore descends to a S^1 action on $\mathbb{CP}^n$. It is easy to check that the induced S^1 action on $\mathbb{CP}^n$ is Hamiltonian and so is equivariantly formal. Moreover, by construction the induced action on $\mathbb{CP}^n$ has only finitely many fixed points such that the restriction of the generalized complex structure $\widetilde{\mathcal{J}}_\epsilon$ to the tangent space of each fixed point is a complex structure.

It then follows from Theorem 5.9 that

$$H^i_{\bar\partial_{\widetilde{\mathcal{J}}_\epsilon}}(M) = 0, \quad if\, i \neq 0.$$

Thus the generalized Hodge number $h^{p,q} = 0$ if $q \neq 0$. In addition, we have

$$\begin{aligned} h^{p,0} &= \dim H^p_{\bar\partial_{\widetilde{\mathcal{J}}_\omega}}(M) \quad \text{(By Theorem 6.1.)} \\ &= \dim H^{n+p}(M). \quad \text{(By Example 2.5.)} \end{aligned}$$

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